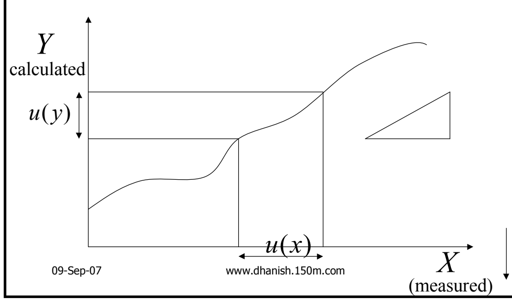
Combined Standard Uncertainty

Uncertainty of a result

- CASE 1: Result Y is a function of a single variable X
- CASE 2a: Result Y is a function of two independent variables X_1, X_2
- CASE 2b: Result Y is a function of several independent variables $X_1, X_2, \ldots X_n$
- CASE 3: Result Y is a function of several variables X_i, some of which are not independent

CASE 1: Result is a function of a single variable, Y=f(X)



Since u(x) is small,

$$u(y) = \frac{dy}{dx} u(x) = c_x u_x$$

$$c_x \equiv \frac{dy}{dx}$$

- Neglecting higher order terms

Example

- The radius of a circle is measured as 100mm with a standard uncertainty of 0.8mm.
- What is the standard uncertainty in the area of the circle?

Solution

A = πr^2 = π*100*100=31415.9266mm^2

$$u_A = c_r u_r$$

$$c_r = \frac{dA}{dr} = 2\pi r = 2*3.14159*100 = 628.3185$$

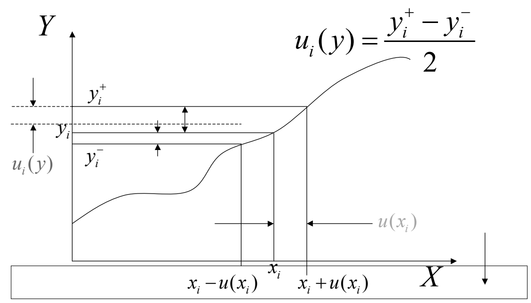
$$u_A = c_r u_r = 628.3185*0.8$$

$$= 502.6548 mm^2$$

Numerical approach

- Calculate y = f(x)
- Calculate $y_i^+ = f(x+u(x_i))$
- Calculate $y_i^- = f(x-u(x_i))$
- Calculate $u_i(y) = (y_i^+ - y_i^-)/2$
- Average of the changes in y due to a change in x
- is the uncertainty in y

Numerical approach



Numerical Approach

$A = \pi r^2 = \pi*100*100 = 31415.9266mm^2$

$A^+ = \pi[r+u(r)]^2 = \pi*100.8*100.8 = 31920.59198$

$A^- = \pi[r-u(r)]^2 = \pi* 99.2*99.2 = 30915.28233$

$u(A) = (|A^+ - A^-|)/2$

$= (31920.59198 - 30915.28233)/2$

$= 502.6548mm^2$

Compare with the linear approximation

CASE 2a: Result Y is a function of two independent variables X

$$Y = f(X_1, X_2)$$

$$u^2(y) = [c_1 u(x_1)]^2 + [c_2 u(x_2)]^2$$

$$c_1 \equiv \frac{\partial f}{\partial x_1} \qquad c_2 \equiv \frac{\partial f}{\partial x_2}$$

Example:

- The radius and height of a cylinder have been measured to be 100mm and 200mm with standard uncertainties of 0.6mm and 0.8mm.
- Determine the standard uncertainty in the volume of the cylinder
- Assume that the measurements are independent

Solution:

- $v = \pi r^2 * h = \pi * 100^2 * 200$
  $= 6283185.3072mm^3$

$$c_r = \frac{\partial V}{\partial r} = 2\pi r h = 2\pi * 100 * 200 = 125663.7061$$

$$c_h = \frac{\partial V}{\partial h} = \pi r^2 = \pi * 100 * 100 = 31415.9265$$

Contd...

$$u^2(v) = [c_r u(r)]^2 + [c_h u(h)]^2$$

$$= [125663.7061 * 0.6]^2 + [31415.9265 * 0.8]^2$$

$$= 6316546812.7531$$

$$u(v) = 79476.7 mm^3$$

Numerical approach

- Calculate y = f(x_1,x_2)
- Calculate y_1^+= f(x_1+u(x_1))
- Calculate y_1^- = f(x_1-u(x_1))
- Calculate u_1(y) = (y_1^+ - y_1^-)/2
- Average of the changes in y due to a change in x_1
- is the uncertainty due to influence x_1

Numerical approach ...

- Calculate y_2^+= f(x_2+u(x_2))
- Calculate y_2^- = f(x_2-u(x_2))
- Calculate u_2(y) = (y_2^+ - y_2^-)/2
- Average of the changes in y due to a change in x_2
- is the uncertainty due to influence x_2

Numerical approach ...

				r+	r-	h+	h-
r	100	u(r)	0.6	100.6	99.4	100	100
h	200	u(h)	0.8	200	200	200.8	199.2
V	6283185			6358810	6208013	6308318	6258053
			ur(V)	75398.22	uh(V)	25132.74	
			uc(V)	79476.71			

CASE 2b: Result Y is a function of N independent variables X

The measurand Y is determined from N other quantities through a functional relationship

$$Y = f(X_1, X_2, ..., X_N)$$

The estimate of X_i is denoted by x_i

All input quantities are independent

$$u_c^2(y) = \sum_{i=1}^{N} [c_i u(x_i)]^2 = \sum_{i=1}^{N} [u_i^2(y)]$$

$$c_i \equiv \frac{\partial f}{\partial x_i} \qquad u_i(y) \equiv |c_i| u(x_i)$$

c_i is the sensitivity coefficient

Example

- A potential difference of 6v is applied to the terminals of a resistor that has a resistance of 2Ω at the defined temperature 20ºC and a linear temperature coefficient of resistance 0.004ºC^{-1}
- The standard uncertainties are 0.2v, 0.1Ω, 0.8ºC and .001ºC^{-1} respectively
- Determine the uncertainty in the power dissipated by the resistance at a temperature of 30ºC

Solution

$$P = \frac{V^2}{R_0[1+\alpha(t-t_0)]}$$

$$= \frac{6^2}{2[1+0.004(30-20)]}$$

$$= 17.3078W$$

Contd...

$$c_v = \frac{2P}{V} = \frac{2*17.3077}{6} = 5.7692$$

$$u_v(P) = c_v u(V)$$

$$= 1.1538W$$

Contd...

$$c_{R_0} = \frac{-P}{R_0} = \frac{-17.3078}{2} = -8.6538$$

$$u_{R_0}(P) = c_{R_0} u(R_0)$$

$$= 0.86538W$$

Contd...

$$c_t = \frac{-P\alpha}{[1+\alpha(t-t_0)]} = \frac{-17.3077*0.004}{[1+0.004(30-20)]}$$

$$= 0.066568$$

$$u_t(P) = c_t u(t) = 0.0532544W$$

Contd...

$$c_\alpha = \frac{-P(t-t_0)}{[1+\alpha(t-t_0)]} = \frac{-17.3077*(30-20)}{[1+0.004(30-20)]}$$

$$= 166.42$$

$$u_\alpha(P) = c_\alpha u(\alpha) = 0.16642W$$

Contd...

$$u_c^2 = u_v^2(P) + u_{R_0}^2(P) + u_t^2(P) + u_\alpha^2(P)$$

$$= 1.1538^2 + 0.86538^2 + 0.05325^2 + 0.16642^2$$

$$= 2.11067$$

$$u_c = 1.4528W$$

Experimental approach

- The sensitivity coefficients can be determined experimentally, measure the change in Y produced by a change in a particular Xi while holding the remaining quantities constant.

Numerical approach

- Calculate $y = f(x_1, x_2, \ldots, x_N)$
- Calculate $y_i^+ = f(x_i + u(x_i))$
- Calculate $y_i^- = f(x_i - u(x_i))$

  for all i=1,.., N
- Calculate $u_i(y) = (y_i^+ - y_i^-)/2$
- Average of the changes in y due to a change in x
- is the uncertainty due to influence i

Finally,

$$u_c(y) = \sqrt{\sum_{i=1}^{N} \left[u_i(y)\right]^2}$$

Example

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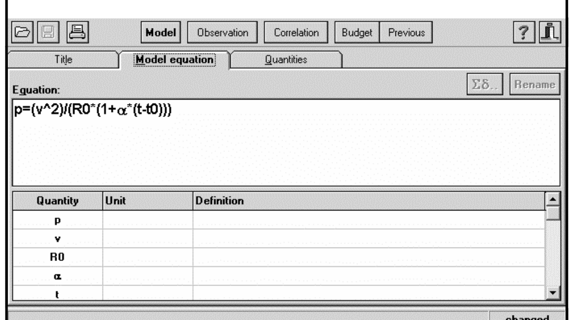
V	R	alpha	t	P	$u_i(y)$
6	2	0.004	30	17.3076923	
6.2	2	0.004	30	18.4807692	
5.8	2	0.004	30	16.1730769	1.153846



Model | Observation | Correlation | **Budget** | Previous

Uncertainty budget:

Quantity	Value	Standard uncertainty	Degrees of freedom	Sensitivity coefficient	Uncertainty contribution	Corr.-coeff.	Index
v	6.000	0.200	50	5.77	1.15	0.79	0.630
R0	2.000	0.100	50	-8.68	-0.868	-0.60	0.356
α	0.00400	0.00100	50	-166	-0.166	-0.11	0.013
t	30.000	0.800	50	-0.0666	-0.0533	-0.04	0.001
t0	20.0						
p	17.31	1.45	95				

Result:
Value: 17.3 ± Expanded uncertainty: 2.9 Coverage factor: 2.0 Coverage: t-table 95%

changed

Problem:

- The mass flow rate through a venturimeter is determined by measuring the time taken to fill a tank with area of cross-section 3mX2m and height 2m. These lengths were determined to an uncertainty +/-0.01m with a triangular distribution.

- Five repetitions gave the times as 50.0, 49.2, 49.0, 50.1, 49.5 which was attributed to the human variation in starting and stopping the stopwatch. The stopwatch used was a digital one of least count 0.2s. The density of the fluid is obtained from a handbook as (800 +/-3)kg/m^3 (99%).
- Determine the mass flow rate and its uncertainty

Model | Observation | Correlation | **Budget** | Previous

Uncertainty budget:

Quantity	Value	Standard uncertainty	Degrees of freedom	Sensitivity coefficient	Uncertainty contribution	Corr.-coeff.	Index
ρ	800.00 kg/cum	1.16 kg/cum	50	0.242	0.282 kg	0.25	0.065
l	3.00000 m	0.00408 m	∞	64.6	0.264 kg	0.24	0.057
b	2.00000 m	0.00408 m	∞	96.9	0.395 kg	0.36	0.128
h	2.00000 m	0.00408 m	∞	96.9	0.395 kg	0.36	0.128
t	49.560 s	0.216 s	4	-3.91	-0.844 kg	-0.76	0.581
δt	0.0 s	0.0577 s	∞	-3.91	-0.226 kg	-0.20	0.042
m	193.70 kg	u_c = 1.11 kg	11				

Result:
Value: 193.7 kg ± Expanded uncertainty: 2.5 kg Coverage factor: 2.3 Coverage: t-table 95%

changed

All input quantities are NOT independent

$$u_c^2(y) = \sum_{i=1}^{N} [c_i u(x_i)]^2$$

$$+ 2\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} c_i c_j u(x_i) u(x_j) r(x_i, x_j)$$

Correlation detailed

$$r(x_i, x_j) = \frac{u(x_i, x_j)}{u(x_i)u(x_j)}$$

is the correlation coefficient between -1 and 1

- $u(x_i,x_j)$ is the covariance between x_i and x_j.
- If x_i and x_j are independent, $r(x_i,x_j)=0$
- OR A change in x_i does not imply a change in x_j

Example

- The two sides of a rectangle are measured as 200mm and 100mm with the same scale with uncertainties of 0.8mm and 0.5mm respectively.
- The two uncertainties are correlated with a correlation coefficient of 0.8
- Determine the uncertainty in area.

Solution

- $A = l*b = 200*100 = 20000mm^2$
- $c_l = b = 100$; $u_l(A) = 100*0.8 = 80mm^2$
- $c_b = l = 200$; $u_b(A) = 200*0.5 = 100mm^2$
- $u_c^2(A) = u_l^2(A) + u_b^2(A) + 2c_lc_bu(l)u(b)r(l,b)$
  $= 6400 + 10000 + 2*100*200*0.8*0.5*0.8 = 29200$
- $u_c(A) = 170.88mm^2$

Numerical approach

				l+	l-	b+	b-	
l	200	u(l)	0.8	200.8	199.2	200	200	
b	100	u(b)	0.5	100	100	100.5	99.5	
A	20000			20080	19920	20100	19900	
			ul(A)	80	ub(A)	100		0.8
				6400		10000	ulb(A)	12800
			uc(A)	170.8801				

Special case

- Where all of the input estimates are correlated with correlation coefficients $r(x_i,x_j)=+1$, the equation reduces to

$$u_c(y) = \sum_{i=1}^{N} c_i u(x_i)$$

Example

- Ten resistors, each of nominal resistance $R_i=1000\Omega$, are calibrated with a negligible uncertainty of comparison in terms of the same 1000Ω standard resistor R_s characterized by a standard uncertainty $u(R_s)=100m\Omega$ as given in its calibration certificate.

Contd...

- The resistors are connected in series with wires having negligible resistance in order to obtain a reference resistance R_{ref} of nominal value $10k\Omega$.
- What is the uncertainty in R_{ref}?

Solution

- Rref = 10*1000 = 10kΩ
- Since $r(x_i,x_j) = r(R_i,R_j) = +1$ for each resistor pair.
- $c_{Rs}(R_{ref}) = 1$
- $u_c(R_{ref})=\Sigma u(R_s) = 10*100m\Omega = 1\Omega$

Thus,

- We can determine the combined standard uncertainty